14.6 Directional Deriv. and Gradient

Entry Task: *(Unit Vector Review)*

If **u** = ⟨ a, b ⟩ is a unit vector, then

- $a^2 + b^2 = 1$
- $a = \cos(\theta), b = \sin(\theta)$

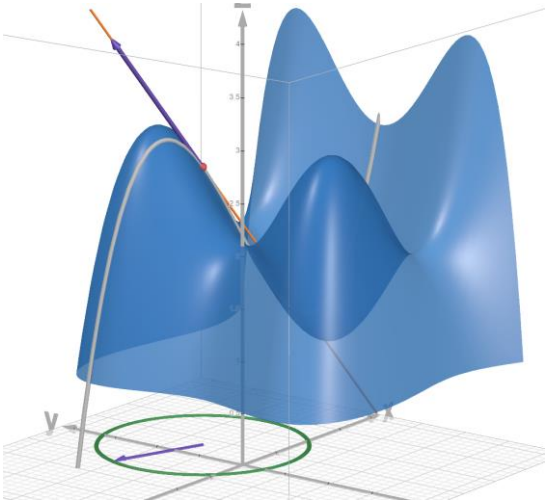
Give several examples of unit vectors that go with "nice" angles on the unit circle:

Suppose you stand at (x_o, y_o) and face in the unit direction $\boldsymbol{u} = \langle a, b \rangle$.

Def'n: Directional derivative

$$D_u f(x_0, y_0)$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



https://www.desmos.com/3d/vgs3ohmid3

Note: $D_{\langle 1,0 \rangle} f = f_x$ and $D_{\langle 0,1 \rangle} f = f_y$

The Chain Rule and Gradient

We can parameterize motion in the direction of $\boldsymbol{u} = \langle a, b \rangle$ using

$$x = x_0 + at,$$

$$y = y_0 + bt,$$

$$z = f(x(t), y(t))$$

What does the chain rule say about $\frac{dz}{dt}$?

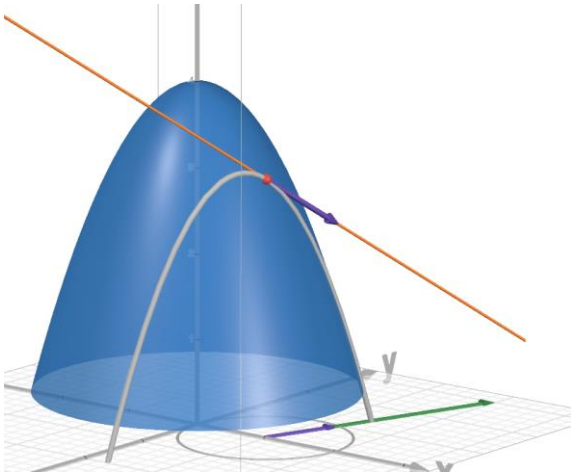
Define the **gradient**:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

From the observation we just made

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$$

Example: *HW Q3–5*



$$f(x,y) = 4 - x^2 - y^2$$

Find the directional derivative at (1,0) in the direction $\boldsymbol{v} = \langle 2, 3 \rangle$.

Step 1: Find ∇f.

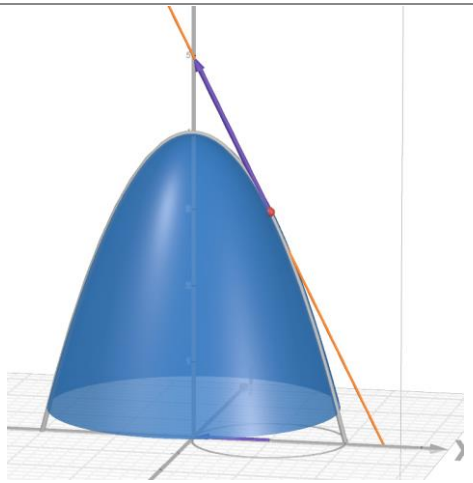
Step 2: Normalize **v**.

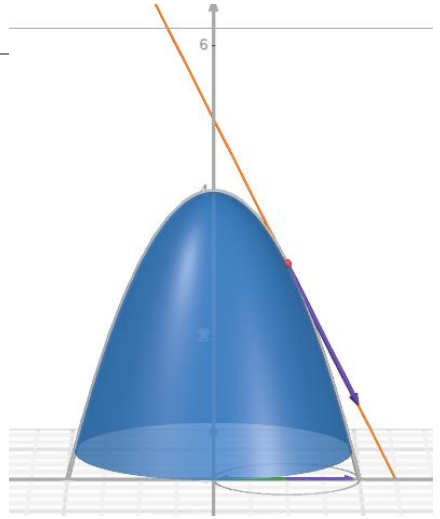
Step 3: Take the dot product.

Gradient Facts: Steepest Increase

Recall the dot-product formula from Ch 12:

$$\nabla f \cdot \boldsymbol{u} = |\nabla f||\boldsymbol{u}| \cos(\theta)$$
$$= |\nabla f| \cos(\theta) \qquad (why?)$$



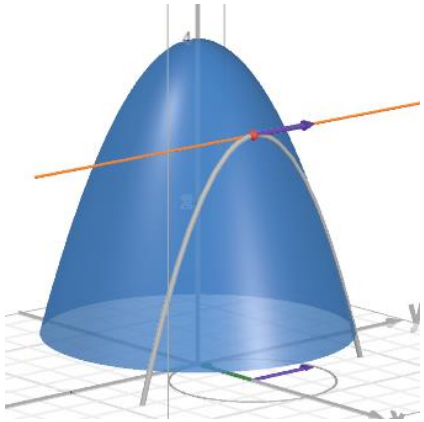


Observations about directional derivative:

Largest when: $\cos(\theta) = 1$

Smallest when: $\cos(\theta) = -1$

Zero when: $\cos(\theta) = 0$



What does this mean….

Key Fact:

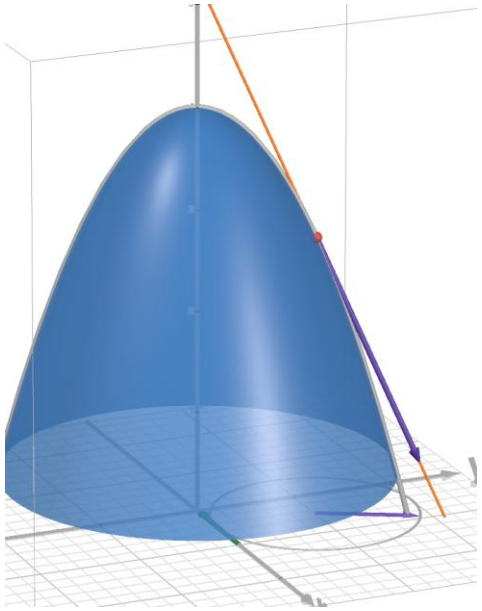
The direction of the gradient vector, ∇f, is the steepest uphill direction. The slope in that direction is $|\nabla f|$

Example: Same surface, new questions

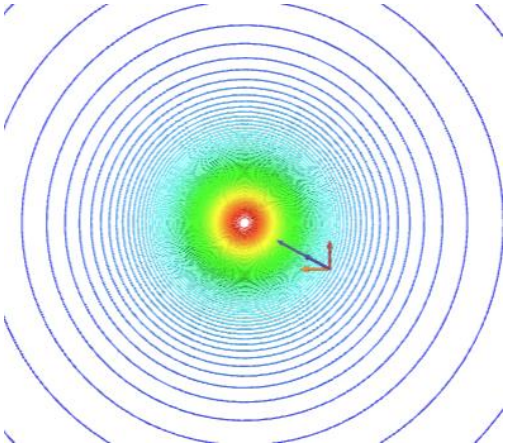
$$f(x,y) = 4 - x^2 - y^2$$

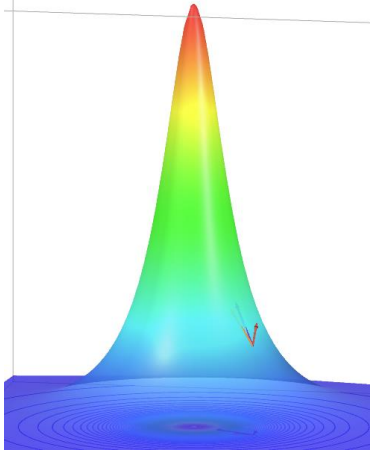
At (1,0), find the direction of greatest increase and the corresponding slope.

At (1/2 ,1), find the direction of greatest decrease and the corresponding slope.



Gradients and Level Curves





Recall the contour maps from 14.1.

$$f(x,y) = k$$

If $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$ stays on a level curve, then $f(x(t), y(t)) = k$.

Differentiating gives (why?)

$$\frac{d}{dt}\left(f\left(x(t)\,, y(t)\right)\right) = 0$$

Thus, $\nabla f \cdot \boldsymbol{r}'(t) = 0$ (why?)

Therefore ∇f is perpendicular to the tangent direction of a regular level curve.

Gradients of Three Variable Functions

We can also define the gradient for a function of three variables:

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

For a regular level surface $f(x, y, z) = k$, the gradient is a normal vector.

Example *HW Q5*

$$f(x, y, z) = x^2 y + y z^2$$

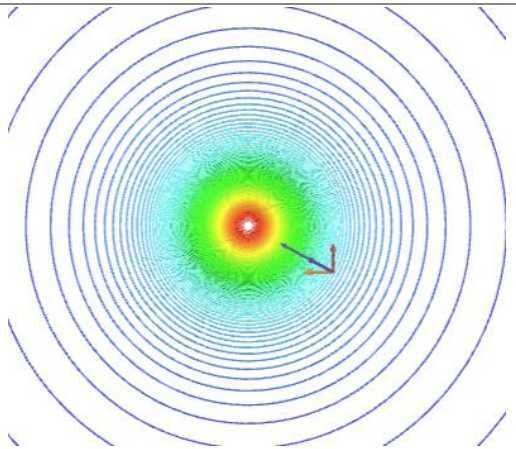
At $(2,7,9)$, find the directional derivative in the direction of $\langle 2, -1,2 \rangle$.

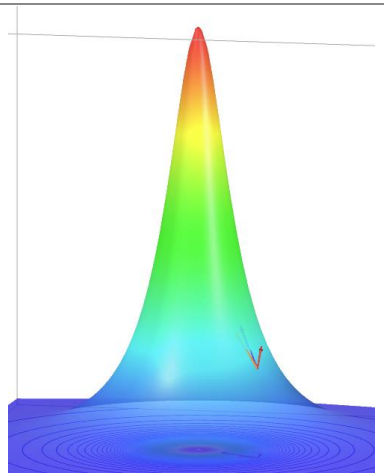
Example:

$$T(x, y, z) = x^2 \sin(yz)$$

At (2,1,π), find the gradient and maximum rate.

Orthogonal Trajectories





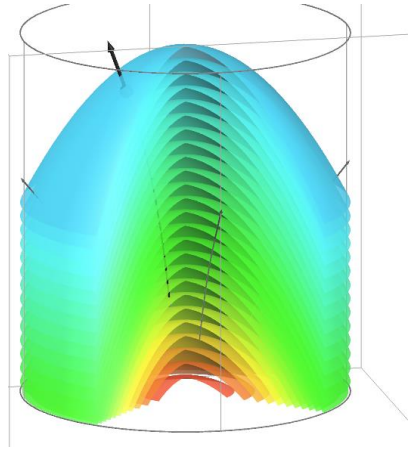
The gradient is perpendicular to a level curve. What happens if we continually walk in the gradient direction?

$$\boldsymbol{r}'(t) = \nabla f(\boldsymbol{r}(t))$$

The resulting paths cross regular contours at right angles.

What if we walk downhill instead of uphill?

$$\boldsymbol{r}'(t) = -\nabla f(\boldsymbol{r}(t))$$



Optional: Gradient Descent

Suppose we want to find a low point of a complicated function.

A simple algorithm:

1. Choose a starting point p_o.
2. Compute the gradient there.
3. Take a small step in the opposite direction.
4. Repeat.

$$p_{n+1} = p_n - \eta \nabla f(p_n)$$

$\eta > 0$ is the step size

Why does this matter?

Gradient descent is used as a tool in optimization and machine learning.